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Commutative Asymptotic Limit of a Quasi-$SU(2)$ Formulation of General Relativity

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The notion of local quasi-gauge bundle structure is introduced. We show that general relativity can be recast in a local quasi-$SU(2)$-bundle framework. In the limit of weak asymptotic gravitational field, this geometrical setup gives rise to spin-2 tensor fields sourcing global charges. If such charges are available, it is shown that the asymptotic geometrical framework is that of a $U(1)$ gauge bundle over S_2, the commutative geometry of the (Dirac) magnetic monopole.

1. INTRODUCTION

In this paper, we want to shed light on the $U(1)\times SU(2)$ aspects of the gravitational interaction. This is an important matter for a better understanding of the unification of interactions of various strengths and ranges.

General relativity will be recast with the help of new spinor variables: canonically conjugate $SU(2)$ Lie-algebra-valued one-forms. These variables suffice, and are more elegant for an initial value formulation of the theory. There consequently arises a noncommutative $[SU(2)]$ setup which is available in the usual nonlinear regime. If the asymptotic behavior of this setup is considered under assumptions of weak gravitational coupling, the above $SU(2)$ variables give rise to $U(1)$ variables and to the commutative geometry of the Dirac monopole provided magnetic (gravitational) charges are allowed not to vanish. The commutative $U(1)$ geometry thus turns out to be an asymptote (at null infinity and in an appropriate sense) of the noncommutative $SU(2)$ geometry. If the asymptotic behavior is allowed in a region of nonlinear coupling, which is connected to null infinity, an $SU(2)\times U(1)$ quasilocal bundle geometry sets in, which encompasses the two regimes (the spacelike and the null-like) in which gravity can be investigated, where the $SU(2)$ geometry relates to the spacelike regime and the $U(1)$ geometry to the null-like regime.

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These Maxwellian aspects of the asymptotic null regime and $SU(2)$ aspects of the spacelike regime (provided nonlinearities are present) bring some clarity to the various facets of the gravitational interaction and, we hope, could be of further use in the understanding of the unification of this interaction with others.

2. LOCAL QUASI-GAUGE BUNDLE STRUCTURE

We first introduce the "deformation" of groups of transformations into quasi groups. Isometry Lie groups will serve as an illustration.

Let us consider the case of isometry Lie algebras on a space-time manifold (M, g_{ab}). Since a Killing vector field (KVF) ξ^a can be characterized by its Killing data at a point p: $(\xi^a, \xi_{ab} = \nabla_a \xi_b)/p$, the commutator $[\xi, \eta]^a = C_p{}^a{}_q \xi^p \eta^q$ of two KVFs ξ^a and η^a is also characterized by its Killing data:

$$(\xi^m \eta^a_m - \eta^m \xi^a_m, \xi^m_a \eta_{mb} - \eta^m_a \xi_{mb} - R_{mnab} \xi^m \eta^n)_p \tag{1}$$

Clearly, the structure constants $C_{pq}{}^a$ of the Killing Lie algebra are completely determined by g_{ab}/p and $R_{abc}{}^d/p$. This suggests that we introduce at p the $n(n+1)/2$-dimensional real vector space V_p of pairs (ξ^a, F_{ab}), where ξ^a is a vector and F_{ab} a skew-symmetric tensor, with the above bracket (1). The bracket operation endows V_p with an algebra structure, but in general this algebra is not associative, nor a Lie algebra—the Jacobi identity will not be satisfied. However, when pairs (ξ, F) are integrable into KVFs, they generate a Lie algebra I_p isomorphic to an isometry Lie algebra of (M, g_{ab}). Requesting that all pairs $(\xi, F)_p$ integrate into a KVF is of course a restrictive condition on the space-time structure. More precisely, it can be shown that

$$\oint_{1,2,3} [[\xi_1, \xi_2], \xi_3] = 0$$

(the Jacobi identity) is satisfied iff

$$M_{ab} = \oint_{1,2,3} \xi_1^m \xi_2^n (R_{mncb\ 3}F_a{}^c - R_{mna}{}^c{}_{3}F_{cb}) + 2\xi_1^{[m} \xi_2^{\eta]} R_{mcab\ 3}F_n{}^c = 0 \tag{2}$$

which is a severe restriction on the space-time curvature. This condition is satisfied in particular for space-times with constant curvature. In that case I_p is isomorphic to the de Sitter Lie algebra, i.e., the $n(n-1)/2$-dimensional Lie algebra of isometries of an n-manifold equipped with a metric of constant curvature, the signature of the metric being the same as that of g_{ab}/p, and the sign of the scalar curvature being the same as that of R/p.

This remark clearly underlines the role of R_{abcd} in the deformation of the algebra structure, and consequently of the Lie group structure of those transformations (of a space-time manifold) which are expected to carry conservation laws. The notion of quasigroups of transformations can cope with such situations. The idea is to describe the "deformation" of a (Lie) group product via a set of real parameters $\theta \equiv (\theta^\alpha)$, $\alpha = 1, 2, \ldots, r$. Thus, a set of transformations (T_θ) will form a continuous quasigroup of transformations if:

1. There exists a unit element T_0, such that

$$x' \equiv [T_\theta(x)]_{\theta=0} = x \qquad \forall x \in (M, g_{ab})$$

2. The modified composition law holds,

$$T'_{\theta'} \circ T_\theta(x) = T_{\Phi(\theta, \theta', x)}(x)$$

implying that the deformation is controlled by Φ and the parameters θ^α.

3. This results in a composition law

$$\Phi^\alpha(\Phi(\theta, \theta'; x), \theta''; x) = \Phi^\alpha(\theta, \Phi(\theta', \theta''; T_\theta(x)); x)$$

4. Here the left and right units coincide:

$$\Phi^\alpha(\theta, 0; x) = \theta^\alpha \qquad \text{and} \qquad \Phi^\alpha(0, \theta'; x) = \theta'^\alpha$$

5. The transformation inverse to T_θ exists:

$$x = T_\theta^{-1}(x')$$

These relations define the right action. The left action is defined similarly.

The generators of these infinitesimal transformations

$$[\Gamma_a = (\partial/\partial\theta)[T_\theta(x')]_{\theta=0}, \qquad \partial/\partial x' = R^i_a\, \partial/\partial x^i$$

obey the following commutation relations:

$$[\Gamma_a, \Gamma_b] = C^d_{ab}(x)\Gamma_d$$

These relations encompass the deformation of the Lie-algebra structure within structure functions, i.e., point-dependent $C_{ab}{}^d$. The resulting modification of the Jacobi identity is expressed by

$$C_{ab}{}^c{}_{,i}R^i_d + C_{da}{}^c{}_{,i}R^i_b + C_{bd}{}^c{}_{,i}R^i_a + C_{ab}{}^e C_{de}{}^c + C_{da}{}^e C_{be}{}^c + C_{bd}{}^c C_{ae}{}^e = 0$$

For an isometry Lie algebra, the above deformation of the identity can be identified with M_{ab}, thus displaying the role of R_{abcd} in the deformation of (Lie) groups into quasigroups.

Below, we shall be concerned with compact Lie groups at infinity. Their deformation in nonasymptotic regions is to be expected. Since such groups are acting on fibers of an asymptotic (principal) fiber bundle, we shall briefly outline the structure of quasi (principal) fiber bundles. The difference here lies in the fact that the Lie algebra, instead of being point-independent and therefore conveniently viewed as a space tangent at the identity of the

Lie group (acting on the fiber), becomes dependent on the point x which projects each fiber on the base space. Let L_a denote the generators of this quasigroup of transformations, and (x, y) a point of the bundle, where x belongs to the base space and y to the fiber. These generators are tangent at the point y of the fiber F_x:

$$L_a = L_a{}^b(x, y)\, \partial/\partial y^b$$

and their commutator is given by

$$[L_a, L_b] = C_{ab}{}^m L_m$$

If, in particular, the structure coefficients $C_{ab}{}^m$ are constants, the usual notion of Lie algebra, and principal fiber bundle, is recovered.

Since the horizontal sections of the bundle must be Lie-derived by the action of the fiber (quasi) group, if T_α defines such a section,

$$[T_\alpha, L_b] = C_{\alpha b}{}^Y T_Y$$

where $C_{\alpha b}{}^Y$ is a tensor field on the bundle, i.e.,

$$C_{\alpha b}{}^Y = C_{\alpha b}{}^Y(x, y)$$

Furthermore,

$$T_\alpha = T_\alpha^\lambda(x, y)\, \partial/\partial x^\lambda + T_\alpha^a(x, y)\, \partial/\partial y^a$$

implies

$$[T_\alpha, T_\beta] = C_{\alpha\beta}{}^\lambda(x, y) T_\lambda + C_{\alpha\beta}{}^a(x, y) L_a$$

3. THE $SU(2)$ GEOMETRY OF GENERAL RELATIVITY

The space-times (M, g_{ab}) under consideration are globally hyperbolic and vacuum. Σ_θ is a Cauchy slice labeled by θ, a time parameter over M, i.e., an affine parameter along a timelike vector field θ^a, so that we have $\theta^a \nabla_a \theta = -1$. Thus, the vector field θ^a can be identified with $\partial/\partial\theta$. Let $N^{AA'}$ denote the vector field normal to Σ_θ, where superscript indices A and A' are $SL(2, \mathbb{C})$ spinor indices on (M, g_{ab}). The $SL(2, \mathbb{C})$ transformations preserving $N_{AA'}$ will be identified with $SU(2)$ attached to Σ_θ. Consider objects of the form $\alpha^{AA'}$ such that (denoting the Hermitian conjugate of α by $\bar{\alpha}$)

$$\bar{\alpha}^{AA'} = -\alpha^{AA'} \qquad \alpha^{AA'} N_{AA'} = 0$$

They clearly generate a real vector space, which is horizontal, i.e., orthogonal to $N_{AA'}$, and is consequently 3-dimensional on $\mathbb{R}$. Such objects are isomorphic to tangent vectors to Σ_θ, the isomorphism being encompassed by soldering forms $\sigma^{b}{}_{AA'}$ such that

$$\sigma^{b}_{AA'}\alpha^{AA'}=\alpha^{b}$$

$SU(2)$ soldering forms are consequently defined by

$$\sigma^{A}_{aB}\equiv q_a^m\sigma_m^{AA'}N_{A'B}$$

Unprimed $SL(2,\mathbb{C})$ spinors tangential to Σ_θ will be identified with $SU(2)$ spinors on Σ_θ.

Objects such as $\sigma_a{}^A{}_B$ on (M, g_{ab}) can be viewed as vertical fields on a local quasibundle B where each fiber is isomorphic to Σ_θ (the base space being coordinatized by θ), valued in the $SU(2)$ Lie algebra attached to Σ_θ.

A covariant derivative can be introduced on $SU(2)$ spinor fields on Σ_θ, which is related to the action of ∇ on $SL(2,\mathbb{C})$ spinor fields on (M, g_{ab}). First a derivative operator is introduced on tensor fields tangential to Σ_θ:

$$D_aT_{b\ldots c}{}^{d\ldots e}=q^i_aq^j_b\ldots q^k_cq^d_m\ldots q^e_n\nabla_iT^{m\ldots n}_{j\ldots k}$$

where $q_a{}^b$ denotes the projection on Σ_θ.

Similarly, a derivation operator is defined on $SU(2)$ spinors on Σ_θ:

$$D_a\alpha_A=q_a{}^b\nabla_b\alpha_A$$

Furthermore,

$$D_{[a}D_{b]}\alpha_A\equiv q^m_aq^n_b\nabla_{[m}\nabla_{n]}\alpha_A$$

This relates the curvature $F_{abA}{}^B$ of D to the spinorial curvature ${}^4R_{abA}{}^B$:

$$F_{abA}{}^B=q_a{}^mq_b{}^n\,{}^4R_{mnA}{}^B$$

Using this setup, it has been shown (Ashtekar, 1988; Sen, 1982) that constraints attached to the embedding of Σ_θ into (M, g_{ab}) and to the vacuum Einstein's equation can be formulated algebraically as follows:

$$\frac{i}{\sqrt{2}}q^b_aG_{bc}N^c=\mathrm{tr}(\sigma^bF_{ab})\equiv\sigma^b{}_A{}^BF_{abB}{}^A$$

$$G_{bc}N^bN^c=\mathrm{tr}(\sigma^a\sigma^bF_{ab})\equiv\sigma^a{}_A{}^B\sigma^b{}_B{}^CF_{abC}{}^A$$

where G_{ab} denotes the Einstein tensor $R_{ab}-\frac{1}{2}Rg_{ab}$.

As a result, constraint equations become vertical polynomials on the quasibundle B. This has interesting consequences in the asymptotic limit. Note that objects such as $\sigma^a{}_A{}^B$ and $F_{abA}{}^B$ are vertical forms on B, valued in the $SU(2)$ Lie algebra. Isomorphisms between fibers of B can be denoted by T_θ, and are provided by the diffeomorphisms generated on (M, g_{ab}) by θ^a. According to the previous section, the $T_\theta'^S$ define a deformation of the geometry attached to Σ_θ as one goes from Σ_θ to $\Sigma_{\theta'}$ in the evolution which carries Cauchy slices into Cauchy slices.

To summarize, the contents of Einstein's equation is now described by a noncommutative $SU(2)$ geometry defined on a quasibundle B. We shall see that in the asymptotic limit, such a geometry gives rise to a commutative geometry attached to a $U(1)$ principal bundle on S_2, with S_2 being diffeomorphic to the boundary of each Σ_θ. We now consider the asymptotic limit of the above spinor formulation of vacuum constraints of general relativity. We shall use the weak-field approximation, ignoring the self-interaction of the field. To that issue, one must linearize the field off a "classical vacuum". Such a vacuum is provided by Minkowski space for which $F_{abA}{}^B$ vanishes. Therefore on every surface of Minkowski space the gauge field $A_{aA}{}^B$ defined by

$$D_a\alpha_M = \partial_a\alpha_M + A_{aM}{}^N\alpha_N$$

is pure gauge and can be set equal to zero without loss of generality. The linearization will be accomplished off flat 3-planes of Minkowski space for which $\sigma_{aA}{}^B$ are flat soldering forms and $A_{aA}{}^B$ is chosen to vanish.

A one-parameter curve of phase space variables $\sigma_{aA}{}^B(\lambda)$, $A_{aA}{}^B(\lambda)$ is introduced near the point $p(\lambda=0)=\mathring{p}\equiv(\mathring{A}_{aA}{}^B=0,\ \sigma_{aA}{}^B)$, and linearized, which leads to the fields

$$h_{aA}{}^B \equiv \frac{d}{d\lambda}\,\sigma_{aA}{}^B(\lambda)\bigg|_{\lambda=0}$$

$$C_{aA}{}^B \equiv \frac{d}{d\lambda}\,A_{aA}{}^B(\lambda)\bigg|_{\lambda=0}$$

Introducing

$$h_{ab} = h_{aA}{}^B\sigma_{bB}{}^A \equiv -\mathrm{tr}(h_a\sigma_b)$$

$$C_{ab} = C_{aA}{}^B\sigma_{bB}{}^A \equiv -\mathrm{tr}(C_a\sigma_b)$$

and taking the derivative of the polynomial constraint equations at $\mathring{p}$, one obtains

$$\partial_a h_a + [C_a, \sigma^a] = 0$$

$$\mathrm{tr}(\sigma^b\partial_{[a}C_{b]}) = 0$$

$$\mathrm{tr}(\sigma^a\sigma^b\,\partial_{[a}C_{b]}) = 0$$

(Recall that $D_a\sigma_{bA}{}^B = 0$, which induces the first of the above equations in the linearization process). Now, it has been shown (Ashtekar, 1988, pp. 124–125) that these linearized constraints in fact reduce to

$$\partial_a h^{ab} = 0 \qquad \partial_a C^{ab} = 0$$

where h^{ab} and C^{ab} denote the traceless part of h^{ab} and C^{ab}.

As a result, the true degrees of freedom of the asymptotic, weakly coupled gravitational field are represented by symmetric, transverse, traceless tensor fields on Σ_θ, which is characteristic of the spin-2 degrees of freedom. This result provides the starting point of the next section.

4. THE COMMUTATIVE ASYMPTOTIC LIMIT

In this section we take the viewpoint that spin-2 fields which emerge in the asymptotic regime from the weakly coupled gravitational field can give rise to global (topological) charges. These charges, in turn, generate a $U(1)$ gauge field which is rather reminiscent of a Maxwellian gauge connection. In this setup, the noncommutative $SU(2)$ geometry gives rise, at infinity, to a commutative $U(1)$ geometry, which suggests that gravitational degrees of freedom, in the presence of suitable underlying space-time topology, could be appropriately described by a $U(1)\times SU(2)$ gauge theory, a result which could shed light on the unification of the gravitational interaction with, e.g., the electromagnetic one, and which, in particular, makes sense in the presence of magnetic monopoles.

Let us introduce, on the space-time manifold, and in the asymptotic region, a standard Newman Penrose null tetrad (l^a, n^a, m_a, ma) with $l\cdot n = -1$ and $m\cdot m = 1$.

Let ε_{abc} denote the canonical orientation on each Cauchy slice Σ_θ. At each point of the space-time manifold, $\varepsilon_{abc}h^{cd}l_d$ and $\varepsilon_{abc}C^{cd}l_d$ define horizontal 2-forms: Ω_{ab} and $^*\Omega_{ab}$. Their introduction is suggested by the fact that spin 2-fields, in the asymptotic null regime, are encompassed by K^{ab} and $^*K^{ab}$, the respective pullbacks, at conformal null-infinity, of $\varepsilon^{amn}\varepsilon^{bpq}\Omega^{-1}C_{mnpq}$ and $\varepsilon^{amn}\varepsilon^{bpq}(\frac{1}{2}\Omega^{-1}\varepsilon_{mn}{}^{rs}C_{rspq})$, where Ω and C_{abcd} denote the conformal metric rescaling scalar field and the Weyl tensor. In the absence of gravitational radiation {vanishing of the Bondi News functions $N_{ab} \equiv$ pullback of $[q_{ac}(R^c_b - \frac{1}{6}R\delta^c_b) - g_{ab}]$}, both K^{ab} and $^*K^{ab}$ are multiples of $n^a n^b$.

We shall now focus on partial Cauchy slices Σ_θ, restricting our attention to asymptotically null slices, i.e., such that in the asymptotic null regime, $N^{AA'}$ coincides with the null vector field n^a. In such a setup, the 2-forms Ω_{ab} and $^*\Omega_{ab}$ can be pulled-back to conformal null-infinity I. The resulting 2-forms are horizontal on the I-bundle, i.e., are the lift, to I, of closed

2-forms ω_{ab} and $^*\omega_{ab}$ defined on the base space of the bundle—the 2-sphere of null generators of I. These closed 2-forms are consequently locally exact, which implies the existence of local potentials for Ω_{ab} and $^*\Omega_{ab}$: $\Omega_{ab} = D_{[a}A_{b]}$ and $^*\Omega_{ab} = D_{[a}{}^*A_{b]}$.

Due to the strong analogy between $\Omega_{ab}(^*\Omega_{ab})$ and the "electric" ("magnetic") part of the spin 2-fields, it is natural to ask whether spin 2-fields can be the source of charges which could be analogous to the electric and magnetic charges in the case of a Maxwell field. This can indeed be accomplished, provided the base space of the I bundle is cohomologically nontrivial (noncontractible base space: an S^2 which can be viewed as the asymptotic limit of noncontractible 2-chains). In that case

$$\int_{S^2} \omega_{ab}\, dS^{ab} \qquad \text{and} \qquad \int_{S^2} {}^*\omega_{ab}\, dS^{ab}$$

define (topological) global charges, which can be viewed as the global mass (magnetic charge) of the space-times under consideration. If such charges are nonvanishing, the I-bundle displays compact fibers [$U(1)$ topology]. On this bundle, potentials A_b and $^*A_b$ define connection 1-forms, valued in the $U(1)$ Lie group, and can be viewed as asymptotic Maxwellian connections which find their origin, in the nonasymptotic regime, in the canonically conjugate fields $\sigma_{aA}{}^B$ and $A_{aA}{}^B$. In the asymptotic decoupling from nonlinearities, $SU(2)$ (noncommutative) gauge degrees of freedom are therefore transformed into $U(1)$ (commutative) degrees of freedom. This can also be viewed as an expression of the dissipation of the general relativity constraints as one goes to the (characteristic) asymptotic null regime, with the conformal null boundary considered as a characteristic initial value surface. We are now ready to consider the nature of the gravitational interaction and its $U(1)\times SU(2)$ features.

In the normal regime, space-time can be considered as an $\mathbb{R}^3$ bundle over $\mathbb{R}$ (i.e., over a timelike curve). In the asymptotic null regime under consideration, timelike orbits are closed, and the structure is that of an $\mathbb{R}^3$ bundle over $U(1)$. If we want to take into account the Lie groups which govern the behavior of the gravitational degrees of freedom, the Lie group structure is rather that of a local quasi-$SU(2)$ bundle over $U(1)$: for each point θ in the $U(1)$ group, one has an $SU(2)$ group acting on the corresponding fiber Σ_θ and related degrees of freedom. In this sense the group $U(1)\times SU(2)$ is governing the gravitational degrees of freedom in an asymptotic region. These degrees are encompassed in the $SU(2)$ Lie-algebra valued (canonically-conjugate)1-forms $\sigma_{aA}{}^B$ and $A_{aA}{}^B$ and in the $U(1)$ Lie-algebra valued (canonically-conjugate) 1-forms A_b and A'_b. These forms describe the $SU(2)\times U(1)$ aspects of the gravitational interaction. Such a result could be useful for the unification of interactions, and sheds light on the role of topological charges such as magnetic charges (Kerner *et al.*, 1989) as far as this issue is concerned.

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